Physics 325: General Relativity
Spring 2019

Problem Set 7

Due: Fri 28 Mar 2019

Reading: Please read Chapter 9 in Hartle.

Problems (continued on pages 2 and 3): For the problems on this problem set, recall that in lecture we discussed two approaches to computing geodesics:

Method I: We express the proper time or length functional as an integral with respect to one of the coordinates. (For example, in 2D polar coordinates, we write $\phi = \phi(r)$ and express the length functional as an integral with respect to $r$.)

Method II: We introduce an arbitrary parameter $\sigma$ along the curve, and express the proper time or length functional as an integral with respect to $\sigma$. (For example, in 2D polar coordinates, we write $\phi = \phi(\sigma)$ and $r = r(\sigma)$.)

1. Geodesics in a (2+1)D slice of the Schwarzschild spacetime. Hartle Problem 8.3.
In this problem we consider the geodesics in a (2 + 1)D spacetime obtained by restricting to the equatorial slice $\theta = 0$ of the Schwarzschild spacetime. As discussed in Chapter 9, the Schwarzschild spacetime describes the geometry outside of a spherically symmetric star or black hole. Note: Although Hartle doesn’t specify which case to consider, please assume that the geodesics are timelike. This problem uses Method II above.

2. Geodesics in Rindler spacetime II. Hartle Problem 8.9. Last week, on Problem Set 6, we apply Method I to computing the timelike geodesics in Rindler spacetime. On this week’s problem set, we will revisit this problem using Method II. For clarity we divide the problem into the following steps:

(a) Determine the geodesic equations by extremizing the action

$$S = -mc^2 \int d\tau = -mc^2 \int \sqrt{-g_{\alpha\beta}(x) \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} \, d\sigma,$$  \hspace{1cm} (1)
in the case of the Rindler metric

\[ ds^2 = g_{\alpha\beta}(x)dx^\alpha dx^\beta = -X^2dT^2 + dX^2. \]  

(2)

Note that after obtaining the Euler-Lagrange equations you will need to convert \( d/d\sigma \) derivatives to \( d/d\tau \) derivatives using

\[ \frac{d}{d\sigma} = \frac{d\tau}{d\sigma} \frac{d}{d\tau} = \sqrt{-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} \frac{d}{d\tau}. \]

You should obtain two equations in the two unknown functions \( T(\tau) \) and \( X(\tau) \). Show that the result is

\[ \frac{d}{d\tau} \left( X^2 \frac{dT}{d\tau} \right) = 0, \]
\[ \frac{d}{d\tau} \left( \frac{dX}{d\tau} \right) + X \left( \frac{dT}{d\tau} \right)^2 = 0, \]

or equivalently

\[ \frac{d^2T}{d\tau^2} + 2 \frac{dT}{d\tau} \frac{dX}{d\tau} = 0, \]
\[ \frac{d^2X}{d\tau^2} + X \frac{dT}{d\tau} \frac{dT}{d\tau} = 0. \]

(b) By comparing the result of part (a) to the general form of the geodesic equation

\[ \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0, \]  

(8.14)

it is possible to read off the nonzero Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \), as we did in lecture. Alternatively, given the metric \( g_{\alpha\beta} \), the Christoffel symbols are determined by

\[ g_{\alpha\delta} \Gamma^\delta_{\beta\gamma} = \frac{1}{2} \left( \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right). \]  

(8.19)

As a check of part (a), use the last equation to compute the nonzero Christoffel symbols in the Rindler metric, and then substitute into Hartle (8.14) to compute the corresponding geodesic equations. Verify that this reproduces the result of part (a).

(c) Directly solve the geodesic equations of part (a) to solve for functions \( T(\tau) \) and \( X(\tau) \). Then eliminate \( \tau \) to determine \( X \) as a function of \( T \). (See hint at end of problem below.)

(d) Finally, instead of directly solving the second order geodesic equations as in part (c), identify a Killing vector \( \xi \). Then, the conservation of \( u \cdot u \) and \( u \cdot \xi \) along the geodesic give two first order equations. Show that these lead to the same results as in part (c).
Hint: Note the similarity between the metric (2) for flat 2D spacetime in Rindler coordinates, and the metric $ds^2 = r^2 d\phi^2 + dr^2$ for flat 2D space in polar coordinates. The two are the same, provided we identify $(r, \phi) = (X, iT)$. Therefore, this problem is almost identical to our analysis in class to find geodesics $\phi = \phi(r)$ in 2D polar coordinates by Method II in lecture, its simplification using the Killing vector approach (Hartle Ex 8.7). The only difference is that some trigonometric functions become hyperbolic trigonometric functions.

3. Gravitational “redshift” for massive particles. Hartle Problem 9.3. Just as a photon loses energy and momentum in escaping a massive star (cf. Sec. 9.2 in Hartle), so too does a massive particle.

Hint: Write $e_0^\alpha = e_t^\alpha = (A, 0, 0, 0)$ and $e_r^\alpha = e_r^\alpha = (0, B, 0, 0)$ in Schwarzschild coordinates $(t, r, \theta, \phi)$, and determine $A, B$ so that these are orthonormal basis vectors. The quantities $E$ and $P$ are components of the momentum 4-vector in the orthonormal basis: $E = e_0 \cdot p$ and $P = e_1 \cdot p$. 